

## NON-ASPHERICAL ENDS AND NONPOSITIVE CURVATURE

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ABSTRACT. We obtain restrictions on the topology of a closed connected manifold  $B$  that bounds a (possibly noncompact) manifold whose interior  $V$  admits a complete Riemannian metric of nonpositive sectional curvature. A sample result is that  $B$  must be aspherical and  $\pi_1$ -incompressible if

- (1)  $V$  has finite volume and  $\pi_1(B)$  is virtually nilpotent, or
- (2)  $\pi_1(B)$  is virtually nilpotent and has no proper torsion-free quotients, or
- (3)  $\pi_1(B)$  is isomorphic to a uniform, irreducible lattice of real rank  $\geq 2$ .

## 1. INTRODUCTION

Throughout the paper all manifolds are smooth, all metrics are Riemannian, the phrase “nonpositive sectional curvature” is abbreviated to “ $K \leq 0$ ”. We say that  $B$  *bounds*  $N$  if  $B$  and  $N$  are connected manifolds such that  $B$  is diffeomorphic to  $\partial N$ ; in this definition  $B$ ,  $N$  need not be compact, so our usage of the term “bound” is non-standard yet we find it technically convenient.

Any aspherical manifold  $B$  bounds a noncompact aspherical manifold, namely  $B \times [0, 1)$ ; in fact, the universal cover of  $B \times (0, 1)$  is a Euclidean space. It is known that  $B \times (0, 1)$  admits a complete metric of  $K \leq 0$  if  $B$  is an infranilmanifold [BK05], or if  $B$  itself admits a complete metric of  $K \leq 0$ . No obstructions are known, e.g. nothing that we know prevents  $B \times (0, 1)$  from carrying a complete metric of  $K \leq 0$ .

If  $B$  is a *closed* aspherical manifold, then for all we know  $B$  could bound a manifold whose interior admits a *finite volume* complete metric of  $K \leq 0$ , and in fact,  $B$  does bound such a manifold when  $B$  is infranil or has a metric of  $K \leq 0$  due to a recent groundbreaking work of Ontaneda [Ont].

This paper studies similar matters when  $B$  is closed and not aspherical. There seem to be no simple description of closed manifolds that bound aspherical ones, and some obstructions and examples are reviewed in Section 2. In Section 3 we discuss results of Gromov [Gro03], Izeki-Nayatani [IN05], and Belegradek [Bel] that lead to examples of closed manifolds that bound aspherical ones but bound

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no manifold whose interior admits a complete metric of  $K \leq 0$ . This theme is developed in Sections 4–5 where new obstructions are found and explored. Section 6 contains a proof of Theorem 4.1, the main technical result of this paper.

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## 2. BOUNDARIES OF ASPHERICAL MANIFOLDS

Before imposing any curvature restrictions we study topological properties of boundaries of aspherical manifolds. Allowing  $B$  to be noncompact almost makes this task meaningless: if  $B$  bounds  $N$ , then any open connected subset  $U$  of  $B$  bounds  $U \cup \text{Int}(N)$ . As we see below aspherical manifolds with compact non-aspherical boundaries also exist in abundance.

A boundary component of a manifold is  $\pi_i$ -*incompressible* if its inclusion is  $\pi_i$ -injective, i.e. induces an injection on the  $i$ th homotopy group, and is *incompressible* if it is  $\pi_i$ -incompressible for each  $i$ . Thus an aspherical manifold has incompressible boundary if and only if each boundary component is aspherical and its inclusion is  $\pi_1$ -injective.

**Example 2.1.** Fiber bundles of aspherical manifolds is aspherical, but their boundary is often not incompressible, e.g. this happens for the product of two compact aspherical manifolds with nonempty boundaries (as is immediate from the cohomological dimension count).

**Example 2.2.** Complete, locally symmetric manifold of  $K \leq 0$  that have finite volume and  $\mathbb{Q}$ -rank  $\geq 3$  are interiors of compact manifolds with non-aspherical boundary.

**Example 2.3.** The fundamental group of any countable, locally finite, finite dimensional cell complex occurs as the fundamental group of the boundary of an aspherical manifold, namely, the boundary of the regular neighborhood of an codimension  $\geq 3$  embedding of the complex into some manifold (e.g. the Euclidean space). If the complex is a manifold, or if it a finite complex, then the boundary is non-aspherical. (The former follows from the homotopy sequence of the normal sphere bundles, and the latter is true because the boundary and the complex have isomorphic fundamental groups, and if the boundary were aspherical, it would be a closed manifold homotopy equivalent to a complex of lower dimension).

**Example 2.4.** Anderson [And87] showed that the total space of any vector bundle over a closed manifold of  $K \leq 0$  admits a complete metric of  $K \leq 0$ . The associated disk bundle has non-aspherical boundary if and only if the fiber dimension is  $\geq 3$ .

**Example 2.5.** There is many a complete manifold  $V$  of  $K \leq 0$  that deformation retracts onto a compact locally convex submanifold  $S$  of codimension 0 (they are usually called *convex-cocompact*). The normal exponential map to  $\partial S$  identifies  $\text{Int}(S)$  with  $V$ , and  $\partial S$  is often not aspherical.

**Example 2.6.** Identifying two aspherical  $n$ -manifolds  $N_1, N_2$  along diffeomorphic, aspherical, compact,  $\pi_1$ -injectively embedded, proper, codimension zero submanifolds  $D_i \subset \partial N_i$  results in an aspherical manifold  $N$ , see [SW79], whose boundary is a union of  $\partial N_1 - \text{Int}(D_1)$  and  $\partial N_2 - \text{Int}(D_2)$  along  $\partial D_1 \cong \partial D_2$ . Considering the case when  $D_i$ 's are disks we conclude that if two manifolds bound aspherical manifolds, then so does their connected sum. The boundary  $\partial N$  is often non-aspherical, e.g. if  $N_1, N_2$  are compact,  $D_i$  is homotopy equivalent to a complex of dimension  $\leq n - 3$ , and the inclusions  $D_i \hookrightarrow \partial N_i$  are not  $\pi_1$ -surjective, then  $\pi_1(\partial N)$  is a nontrivial amalgamated product over  $\pi_1(D_i)$ , so  $\partial N$  is not aspherical by a Mayer-Vietoris argument in group cohomology.

With plenty of examples, we now turn to obstructions. In order for  $B$  to bound an aspherical manifold, a certain covering space of  $B$  must bound a contractible manifold. In formalizing how this restricts the topology of  $B$ , the following definition is helpful: given a class of groups  $\mathcal{Q}$ , a group is *anti- $\mathcal{Q}$*  if it admits no nontrivial homomorphism into a group in  $\mathcal{Q}$ . Clearly, the class of anti- $\mathcal{Q}$  groups is closed under extensions, quotients, and any group generated by a family of anti- $\mathcal{Q}$  subgroups is anti- $\mathcal{Q}$ .

**Example 2.7.** Let  $\mathcal{A}_n$  denote the class of fundamental groups of aspherical  $n$ -manifolds. Here are some examples of anti- $\mathcal{A}_n$  groups:

- (1) Any group generated by a set of finite order elements.
- (2) Any non-torsion-free simple group.
- (3) Any irreducible lattice in the isometry group of a symmetric space of rank  $\geq 2$  and dimension  $> n$  [BF02].
- (4) Certain finitely presented groups with strong fixed point properties, see [ABJ<sup>+</sup>09, Theorem 1.5].
- (5) Any non-elementary finitely presented relatively hyperbolic group has a finitely presented anti- $\mathcal{A}_n$  quotient, obtainable by adding finitely many relators [AMO07, Corollary 1.6].
- (6) Any finitely presented anti- $\mathcal{A}_n$  group is a quotient of an anti- $\mathcal{A}_n$  non-elementary hyperbolic group [BO08].

The following summarizes some obstructions that prevent a manifold from bounding an aspherical one.

**Theorem 2.8.** *If  $B$  bounds an aspherical, non-contractible  $n$ -manifold, and  $\pi_1(B)$  is anti- $\mathcal{A}_n$ , then  $B$  is noncompact, parallelizable, its  $\mathbb{Z}$ -valued intersection form vanishes, and its  $\mathbb{Q}/\mathbb{Z}$ -valued torsion linking form vanishes.*

*Proof.* Since  $\pi_1(B)$  is anti- $\mathcal{A}_n$ , the manifold  $B$  also bounds a contractible manifold  $W$ , whose interior is the universal cover of  $\text{Int}(N)$ . Since  $W$  is parallelizable,  $B$  is stably parallelizable, and in particular orientable.

Let us show that  $B$  is noncompact. The long exact sequence of the pair gives a surjection  $H_n(W, B) \rightarrow H_{n-1}(B)$ , so if  $B$  were compact, Poincaré duality would imply nontriviality of  $H_{n-1}(B)$ , and hence nontriviality of  $H_n(W, B) \cong H_c^0(W)$ , so there is a compactly supported constant function on the 0-skeleton of  $W$  which is only possible if  $W$  were compact, but by assumptions  $\pi_1(N)$  is nontrivial, and hence infinite, so  $W$  is noncompact.

Since  $B$  is non-compact, its stable parallelizability implies parallelizability.

Recall that the intersection form  $H_k(B) \times H_{n-1-k}(B) \rightarrow \mathbb{Z}$  can be computed via algebraic intersection numbers. If  $\alpha, \beta$  are simplicial cycles in  $B$  of complementary dimensions, then since  $B$  bounds a contractible manifold  $W$ , there are simplicial chains  $\hat{\alpha}, \hat{\beta}$  in  $W$  with  $\partial\hat{\alpha} = \alpha$  and  $\partial\hat{\beta} = \beta$ . Choosing  $\hat{\alpha}, \hat{\beta}$  transverse, and subdividing if necessary, one observes that the intersection of  $\hat{\alpha}, \hat{\beta}$  is a 1-chain  $\hat{c}$ , and the 0-homology class of  $\partial\hat{c}$  equals the algebraic intersection number  $\iota(\alpha, \beta)$  in  $B$ , so that  $\iota(\alpha, \beta) = 0$ .

If  $\tau H_k(B)$  denotes the torsion subgroup of  $H_k(B)$ , then the linking form  $\text{lk}: \tau H_k(B) \times \tau H_{n-2-k}(B) \rightarrow \mathbb{Q}/\mathbb{Z}$  is defined as follows: given two torsion cycles  $\alpha, \beta$  the linking coefficient  $\text{lk}([\alpha], [\beta])$  equals  $\frac{1}{m}\iota(z, \beta)$  modulo 1, where  $z$  is any chain in  $B$  with  $\partial z = m\alpha$ . If  $\hat{\beta}$  is a chain in  $W$  with  $\partial\hat{\beta} = \beta$ , then  $\iota(z, \beta)$  equals the intersection number of  $z$  and  $\hat{\beta}$  in  $W$ . Since  $\alpha = \partial\hat{\alpha}$  in  $W$ , we conclude that  $z - m\hat{\alpha}$  is a cycle in  $W$ , which is a boundary, as  $W$  is contractible. The intersection number of any boundary with  $\hat{\beta}$  is zero, hence  $\iota(z, \beta)$  is divisible by  $m$ , so  $\text{lk}([\alpha], [\beta]) = 0$ .  $\square$

**Remark 2.9.** The argument in the beginning of the above proof shows that if a closed  $(n-1)$ -manifold  $B$  bounds a manifold  $W$  such that  $H_{n-1}(W; \mathbb{Z}_2) = 0$ , then  $W$  is compact.

**Remark 2.10.** The same proof gives that if  $B$  bounds a contractible manifold, then  $B$  is stably parallelizable, and its intersection form and torsion linking form vanish. By Remark 2.9 if  $B$  is compact, then it equals the boundary of a compact contractible manifold, so  $B$  is a homology sphere, and conversely, any homology sphere bounds a topological contractible manifold [Ker69, Fre82].

**Example 2.11.** The following manifolds do not bound aspherical ones:

- (1) The connected sum of lens spaces, because it is a closed manifold whose fundamental group is anti- $\mathcal{A}_n$ .
- (2) The product of any manifold with  $CP^k$  with  $k \geq 2$  because it contains a simply-connected open subset which is not parallelizable, namely, the tubular neighborhood of  $CP^k$ .

- (3) The connected sum of any manifold and the product of two closed manifolds whose fundamental groups are anti- $\mathcal{A}_n$ , as the slice inclusions of the factors in the product have nonzero intersection number.
- (4) The product of a punctured 3-dimensional lens space and a closed manifold whose fundamental group is anti- $\mathcal{A}_n$  (for if  $\alpha$  represents a generator in the first homology of the lens space, then it links nontrivially with itself, see e.g. [PY03], so its product with the closed manifold factor links nontrivially with  $\alpha$ ).
- (5) Any manifold that contains the manifold in (2), (3), or (4) as an open subset.

### 3. REDUCTIVE GROUPS AND REGULAR NEIGHBORHOODS

Let  $\mathcal{NP}_n$  denote the class of the fundamental groups of complete  $n$ -manifolds of  $K \leq 0$ . By Cartan-Hadamard theorem, any complete manifold of  $K \leq 0$  is covered by a Euclidean space, so that  $\mathcal{NP}_n$  is a subset of  $\mathcal{A}_n$ . The two classes coincide for  $n = 2$  by the uniformization and classification of surfaces. On the other hand, for each  $n \geq 4$  there is a closed, locally CAT(0) and hence aspherical,  $n$ -manifold, whose fundamental group is not in  $\mathcal{NP}_n$  [DJ91, DJL12].

Gromov [Gro03] (cf. [NS11, Theorem 1.1]) and Izeki-Nayatani [IN05] showed that there is many a finite aspherical complex  $Y$  such that  $\pi_1(Y)$  is anti- $\mathcal{NP}_n$  for all  $n$ , which immediately implies:

**Theorem 3.1.** *There is a closed non-aspherical manifold that*

- (i) *bounds a manifold whose interior is covered by a Euclidean space;*
- (ii) *bounds no manifold whose interior has a complete metric of  $K \leq 0$ .*

The manifold is obtained by thickening  $Y$  to a compact aspherical manifold  $N$  chosen so that  $Y$  has codimension  $\geq 3$  in  $N$ , which ensures that  $\pi_1(\partial N) \cong \pi_1(Y)$ , the universal cover of  $\text{Int}(N)$  is a Euclidean space, and  $\partial N$  is non-aspherical. Izeki-Nayatani give explicit  $Y$ 's, such as the quotient of the Euclidean buildings by a uniform lattice in a  $p$ -adic symmetric space, while Gromov establishes existence of  $Y$  with (Gromov) hyperbolic  $\pi_1(Y)$ , but to date there is no explicit example of such a  $Y$ .

A group  $G$  *reductive* if for any epimorphism  $G \rightarrow H$  such that  $H$  is a discrete, non-cocompact, torsion-free isometry group of a Hadamard manifold, the group  $H$  stabilizes a horoball or acts cocompactly on a totally geodesic, proper submanifold.

**Example 3.2.** (1) Clearly, the property of being reductive is inherited by quotients, and every group that is anti- $\mathcal{NP}_n$  for all  $n$  is reductive.

(2) Any finitely generated, virtually nilpotent group is reductive, see [Bel] where this is deduced by combining results of Gromov [BGS85] with the Flat Torus Theorem.

(3) Any irreducible, uniform lattice in the isometry group of a symmetric space of  $K \leq 0$  and real rank  $> 1$  is reductive, see [Bel] where it is deduced from the harmonic map superrigidity.

**Remark 3.3.** If  $G$  is a group as in Examples 3.2 (2), (3), then any torsion-free, quotient of  $G$  is the fundamental group of a closed aspherical  $m$ -manifold, which is an infranil or irreducible, locally symmetric of  $K \leq 0$  of rank  $\geq 2$ , respectively, where  $m$  is bounded above by the virtual cohomological dimension of  $G$ . If  $G$  is virtually nilpotent this follows from [DI94], and when  $G$  is a higher rank lattice one invokes Margulis's Normal Subgroup Theorem.

The following definition helps combine the above examples of reductive groups: Given groups  $I, J$  and a class of groups  $\mathcal{Q}$  we say that  $I$  *reduces to  $J$  relative to  $\mathcal{Q}$*  if every homomorphism  $I \rightarrow Q$  with  $Q \in \mathcal{Q}$  factors as a composite of an epimorphism  $I \rightarrow J$  and a homomorphism  $J \rightarrow Q$ . Clearly if  $Q = \mathcal{NP}_n$ , and  $J$  is reductive, then so is  $I$ , and moreover,  $I$  and  $J$  have the same quotients in  $\mathcal{NP}_n$ .

**Remark 3.4.** The class of groups that reduce to  $J$  relative to  $\mathcal{Q}$  is sizable, e.g. if  $K$  is anti- $\mathcal{Q}$ , then the direct product  $K \times J$  reduces to  $J$  relative to  $\mathcal{Q}$ ; more generally, the same is true for any amalgamated product obtained by identifying a subgroup  $A \leq J$  with the second factor of  $K \times A$ . Furthermore, a variant of the Rips construction established in [BO08] implies that if there is an anti- $\mathcal{Q}$  non-elementary hyperbolic group, then for any finitely presented group  $J$ , there is a non-elementary hyperbolic group  $I$  that reduces to  $J$  relative to  $\mathcal{Q}$ . This result applies to  $\mathcal{Q} = \mathcal{NP}_n$  since  $\mathbb{Z}_2 * \mathbb{Z}_3$  is anti- $\mathcal{NP}_n$  and non-elementary hyperbolic (if instead of  $\mathbb{Z}_2 * \mathbb{Z}_3$  one uses the anti- $\mathcal{NP}_n$ , torsion-free, hyperbolic groups of [Gro03], then  $I$  becomes torsion-free).

With the above terminology the main result of [Bel] and Remark 2.9 immediately implies:

**Theorem 3.5.** *Let  $B$  be a closed  $(n-1)$ -manifold such that  $\pi_1(B)$  is reductive and any nontrivial quotient of  $\pi_1(B)$  in the class  $\mathcal{NP}_n$  has a finite classifying space of dimension  $\leq n-3$ . If  $B$  bounds a manifold  $N$  such that  $\text{Int}(N)$  admits a complete metric of  $K \leq 0$ , then  $N$  is compact, and the inclusion  $B \rightarrow N$  is a  $\pi_1$ -isomorphism.*

Stallings's embedding up to homotopy type theorem [Sta] implies that  $N$  of Theorem 3.5 deformation retracts to a regular neighborhood of a finite subcomplex of codimension  $q \geq 3$ , so an excision argument as in [Sie69, Theorem

2.1] shows that  $N$  is obtained by attaching an h-cobordism to the boundary of the regular neighborhood.

Stating the next theorem required some background. Let  $N$  be a compact  $n$ -manifold that is homotopy equivalent to a closed  $m$ -manifold  $M$  such that  $q = n - m \geq 3$ , and the inclusion  $\partial N \rightarrow N$  is a  $\pi_1$ -isomorphism. The Browder-Casson-Haefliger-Sullivan-Wall embedding theorem [Wal99, Corollary 11.3.4] shows that  $N$  is obtained by attaching an h-cobordism to the boundary of the regular neighborhood of a PL-embedded copy of  $M$ . Abstract regular neighborhoods of  $M$  of codimension  $q \geq 3$  are  $q$ -block bundles over  $M$ , and they are classified by homotopy classes of maps of  $M$  into  $\widetilde{BPL}_q$ , the classifying space of the semisimplicial group  $\widetilde{PL}_q$  [RS68a]. Taking product with the  $(q-k)$ -cube defines a stabilization map  $\widetilde{BPL}_k \rightarrow \widetilde{BPL}_q$ . Each  $q$ -block bundle  $R$  over  $M$  defines a spherical fibration over  $M$  with structure group in  $\widetilde{PL}_q$  and fiber  $S^{q-1}$ ; up to homotopy the fibration is the inclusion  $\partial R \hookrightarrow R$  [RS68b]. Spherical fibrations over  $M$  with fiber  $S^{q-1}$  are classified by maps  $M \rightarrow BG_q$ , and we say that *the structure group of a fibration reduces to  $\widetilde{PL}_{k \leq q}$*  if its classifying map factors (up to homotopy) through  $\widetilde{BPL}_k \rightarrow \widetilde{BPL}_q \rightarrow BG_q$ . A spherical fibration is *linear* if it is isomorphic to a unit sphere bundle of a vector bundle.

**Theorem 3.6.** *Let  $M$  be a closed  $m$ -manifold that is either infranil or an irreducible, locally symmetric of  $K \leq 0$  and real rank  $\geq 2$ . Let  $B$  be a closed  $(n-1)$ -manifold such that  $q = n - m \geq 3$ , and  $\pi_1(B)$  reduces to  $\pi_1(M)$  relative to  $\mathcal{NP}_n$ . If  $B$  bounds a manifold whose interior admits a complete metric of  $K \leq 0$ , then*

- (1)  *$B$  is the total space of a spherical fibration  $\beta$  over  $M$  whose structure group reduces to  $\widetilde{PL}_q$ .*
- (2) *If  $q \geq 4$  and the structure group of  $\beta$  does not reduce to  $\widetilde{PL}_{q-1}$ , or if  $q = 3$  and  $\beta$  has no cross-section, then  $M$  admits a metric of  $K \leq 0$  and  $\beta$  is linear.*

*Proof.* Any torsion-free quotient of  $\pi_1(B)$  is also a quotient of  $\pi_1(M)$  which by Remark 3.3 is the fundamental group of a closed aspherical manifold of dimension  $\leq m$ . Thus Theorem 3.5 applies.

If  $N$  is as in Theorem 3.5, then the isomorphism  $\pi_1(B) \rightarrow \pi_1(N)$  factors through a surjection  $\pi_1(B) \rightarrow \pi_1(M)$ , so  $N$  is homotopy equivalent to  $M$ , and by the preceding discussion  $N$  is obtained by attaching an h-cobordism to the boundary of the regular neighborhood of a PL-embedded copy of  $M$ , which proves (1). (Actually, the h-cobordism is trivial smoothly if  $n \neq 4, 5$  and topologically if  $n = 4, 5$ , but we do not use this).

Let  $H \cong \pi_1(M)$  be the deck-transformation isometry group of the nonpositively curved metric on the universal cover of  $\text{Int}(N)$ .

If  $H$  stabilizes a horoball bounded by a horosphere  $S$ , then  $\text{Int}(N)$  is diffeomorphic to  $\mathbb{R} \times S/H$ . In particular, the associated spherical fibration has a section obtained by sliding along the  $\mathbb{R}$ -factor. If  $q \geq 4$ , then again the homotopy equivalence  $M \rightarrow S/H$  is homotopic to a PL-embedding [Wal99, Corollary 11.3.4], whose regular neighborhood is a block bundle with structure group in  $\widetilde{PL}_{q-1}$ , and hence the same is true for its product with  $\mathbb{R}$ , so that the structure group of the associated  $S^{q-1}$ -fibration reduces to  $\widetilde{PL}_{q-1}$ .

If  $H$  does not stabilize a horoball, then by superrigidity or the Flat Torus Theorem,  $H$  acts cocompactly on a totally geodesic subspace whose  $H$ -quotient is diffeomorphic to  $M$  (and in fact homothetic to  $M$  if it has higher rank, or affinely equivalent if  $M$  is flat). The normal exponential map to the subspace is an  $H$ -equivariant diffeomorphism, which identifies  $\text{Int}(N)$  with the normal bundle to  $M$ . Thus the associated spherical fibration is linear.  $\square$

**Example 3.7.** In each of the following cases  $B$  bounds a manifold covered by a Euclidean space, but bounds no manifold whose interior admits a complete metric of  $K \leq 0$ :

- (1)  $B$  is the total space of a linear  $S^{q-1}$  linear bundle over  $M$  with no cross-section, where  $q \geq 3$  and  $M$  is a non-flat infranilmanifold (e.g. if  $M$  is orientable, even-dimensional, non-flat infranilmanifold, then the pullback of the unit tangent bundle under a degree one map  $M \rightarrow S^m$  has nonzero Euler class, and hence no cross-section).
- (2)  $B = M \times \partial C$  where  $C$  be a compact contractible manifold such that  $\pi_1(\partial C)$  is a nontrivial group generated by finite order elements. (In each dimension  $\geq 5$  there are infinitely many such  $C$ 's, namely, given any finitely presented superperfect group  $K$  there is  $C$  with  $\pi_1(\partial C) \cong K$  [Ker69], and the desired infinite family is obtained by varying  $K$  among the free products of finitely many superperfect finite groups.)
- (3)  $B$  is the boundary of a  $q$ -block bundle over  $M$  such that  $q \geq 4$ , the associated spherical fibration is not linear, and its structure group does not reduce to  $\widetilde{PL}_{q-1}$ . (Such block bundles exist below metastable range; here is a specific example where  $q$  is even, and  $m$  is large enough. It is well-known but apparently not recorded in the literature cf. [Kle], that for  $q \geq 3$  the classifying space  $B\widetilde{PL}_q$  is rationally equivalent to  $BO \times BG_q$ , which can be deduced by combining the following results:  $(\widetilde{PL}, \widetilde{PL}_q) \rightarrow (G, G_q)$  is a weak homotopy equivalence [RS68b, Theorem 1.11],  $G$  is rationally contractible [MM79, Chapter 3A], and  $BO \rightarrow B\widetilde{PL}$  is a rational equivalence, see [MM79, Chapter 4C] and [RS68b, Corollary 5.5]. Now  $BO$  is rationally the product of Eilenberg-MacLane spaces corresponding to Pontryagin classes, and if  $q$  is even, then  $BG_q$  is rationally  $K(\mathbb{Q}, q)$ , detected by the Euler class. If  $m$  is large enough,



there exist cohomology classes  $x, y \in H^*(M)$  of degrees  $q, 4i$  with  $i > \frac{q}{2}$ . As in [BK03, Appendix B] one can realize nonzero multiples of  $x, y$  as the Euler class  $e$  and the Pontryagin class  $p_i$  of a block bundle over  $M$ . Its structure group does not reduce to  $\widehat{PL}_{q-1}$  as  $e \neq 0$ , so the associated spherical fibration has no cross-section. The fibration is nonlinear because for linear bundles  $p_i = 0$  for  $i > \frac{q}{2}$ . Another example can be obtained if  $y$  has degree  $2q$  and is not proportional to  $x^2$  by realizing a multiple of  $y$  as  $p_{q/2}$  (using that  $e^2 = p_{q/2}$  are equal for linear bundles.)

#### 4. ENDS WITH INJECTIVITY RADIUS GOING TO ZERO

Following [BGS85], we say that a subset  $S$  of a Riemannian manifold *has*  $\text{Inj Rad} \rightarrow 0$  if and only if for every  $\varepsilon > 0$  the set of points of  $S$  with injectivity radius  $\geq \varepsilon$  is compact; otherwise,  $S$  *has*  $\text{Inj Rad} \not\rightarrow 0$ . For example, by volume comparison any finite volume complete manifold of  $K \leq 0$  has  $\text{Inj Rad} \rightarrow 0$ . Schroeder proved [BGS85, Appendix 2] that any complete manifold of  $-1 \leq K \leq 0$  and  $\text{Inj Rad} \rightarrow 0$  is the interior of a compact manifold with boundary provided it contains no sequence of totally geodesic, immersed, flat 2-tori whose diameters tend to zero.

Given a compact boundary component  $B$  of a manifold  $N$ , an *end*  $E$  of  $\text{Int}(N)$  that corresponds to  $B$  is the intersection of  $\text{Int}(N)$  with a closed collar neighborhood of  $B$ ; note that  $E$  is diffeomorphic to  $[1, \infty) \times B$ .

The *cohomological dimension* of a group  $G$  is denoted  $\text{cd}(G)$ .

Here is the main result of this paper whose proof is in Section 6.

**Theorem 4.1.** *Let  $N$  be a manifold with compact connected boundary  $B$ , let  $E$  be an end of  $V := \text{Int}(N)$  that corresponds to  $B$ , and let  $H$  be the deck-transformation group of the universal cover  $\tilde{V} \rightarrow V$  corresponding to the image of the inclusion induced homomorphism  $\pi_1(E) \rightarrow \pi_1(V)$ . If  $\tilde{V}$  admits a complete  $H$ -invariant metric  $g$  of  $K \leq 0$  and  $\tilde{V}$  contains an  $H$ -invariant horoball, then*

- (1)  $\dim(B) = \text{cd}(H)$  if and only if  $B$  is incompressible in  $N$ .
- (2) If  $\text{Inj Rad} \rightarrow 0$  on  $E$ , then  $B$  is incompressible in  $N$ .
- (3) If  $B$  is incompressible in  $N$ , then the universal cover of  $B$  is homeomorphic to a Euclidean space.

**Corollary 4.2.** *Let  $B$  be a closed manifold that bounds a manifold  $N$ , and let  $E$  be an end of  $\text{Int}(N)$  corresponding to  $B$ . If  $\pi_1(B)$  is reductive, and  $\text{Int}(N)$  admits a complete metric of  $K \leq 0$  and  $\text{Inj Rad} \rightarrow 0$  on  $E$ , then  $B$  is incompressible in  $N$ .*

*Proof.* If  $H$  is as in Theorem 4.1, then since  $H$  is reductive, it either stabilizes a horoball or acts cocompactly on a totally geodesic proper submanifold  $S$ . In the former case  $B$  is incompressible by Theorem 4.1. The latter case cannot happen because the nearest point projection onto  $S$  is  $H$ -equivariant and distance nonincreasing, and there is a lower bound for displacement of elements of  $H$  on  $S$  while the assumption  $\text{Inj Rad} \rightarrow 0$  on  $E$  gives a sequence of points of  $V$  whose displacements under some elements of  $H$  tend to zero.  $\square$

**Example 4.3.** If  $B$  is the total space of a linear  $S^k$  bundle with  $k \geq 2$  over a manifold  $M$  as in 3.6, then  $B$  bounds the associated disk bundle, whose universal cover is the Euclidean space, but  $B$  does not bound a manifold whose interior admits a complete metric of  $K \leq 0$  and  $\text{Inj Rad} \rightarrow 0$ .

## 5. ENDS WITH FUNDAMENTAL GROUPS OF CODIMENSION ONE

A small variation on the proof of Theorem 4.1 yields:

**Addendum 5.1.** *The part (1) of Theorem 4.1 holds if the assumption “ $\tilde{V}$  contains an  $H$ -invariant horoball” is replaced by “ $\tilde{V}$  contains an  $H$ -invariant totally geodesic submanifold of codimension one”.*

**Corollary 5.2.** *Let  $B$  be a closed  $(n-1)$ -manifold such that  $\pi_1(B)$  is reductive and any nontrivial quotient of  $\pi_1(B)$  in the class  $\mathcal{NP}_n$  has cohomological dimension  $n-1$ . If  $B$  bounds a manifold  $N$  such that  $\text{Int}(N)$  admits a complete metric of  $K \leq 0$ , then  $B$  is incompressible in  $N$ .*

*Proof.* If  $H$  is as in Theorem 4.1, then  $H$  cannot be trivial by Theorem 2.8, hence  $\text{cd}(H) = \dim(B)$ , so  $B$  is incompressible in  $N$  by Addendum 5.1.  $\square$

**Example 5.3.** Corollary 5.2 applies to the manifold  $M$  as in Theorem 3.6 provided  $\pi_1(M)$  has no proper torsion-free quotient, which ensures that the cohomological dimension of any nontrivial torsion-free quotient of  $\pi_1(M)$  equals  $\dim(M)$ . In fact if  $M$  is higher rank, irreducible, locally symmetric manifold of  $K \leq 0$ , then  $\pi_1(M)$  has no proper torsion-free quotients by the Margulis Normal Subgroup Theorem. There are also infranilmanifolds whose fundamental group has no proper torsion-free quotients. This includes all 3-dimensional infranilmanifolds with zero first Betti number, such as the infinite family 2 of [DIKL95, page 156] and the Hantzsche-Wendt flat 3-manifold, as well as some higher-dimensional flat manifolds in dimensions  $p^2 - 1$  [GS99, Theorem 1(9)] and  $p^3 - p$  [Cid02, page 29] for any prime  $p$ .

**Remark 5.4.** If one is not concerned with making sure that  $B$  bounds a manifold whose interior is covered by  $\mathbb{R}^n$ , there is a simple recipe for constructing  $B$ 's to which Corollaries 4.2 and 5.2 apply; for concreteness we focus on the latter. Start with  $\pi_1(M)$  as in Example 5.3, consider any finitely presented

group that reduces to  $\pi_1(M)$  relative to  $\mathcal{NP}_n$ , and realize the group as the fundamental group of a closed  $(n-1)$ -manifold  $B$ , which is always possible for  $n \geq 5$ . Now if  $B$  does bound a manifold whose interior admits a complete metric of  $K \leq 0$ , then Corollary 5.2 forces  $B$  to be incompressible, and hence aspherical with  $\pi_1(B) \cong \pi_1(M)$ . For instance,  $B$  is not incompressible if

- $\pi_1(B)$  has a nontrivial anti- $\mathcal{NP}_n$  subgroup, or
- $B$  is the connected sum  $M \# S$ , where  $S$  is simply-connected and not a homotopy sphere (if  $M \# S$  were aspherical, its universal cover would contain a codimension one sphere bounding a punctured copy of  $S$ , so the latter would be contractible).

It takes more effort to find  $B$  that bounds an manifold whose universal cover is a Euclidean space but which cannot bound a manifold whose interior admits a complete metric of  $K \leq 0$ . Here is an infinite family of such examples.

**Example 5.5.** Let  $C$  be a compact contractible manifold as in Example 3.7(3). Fix a closed embedded disk  $\Delta \subset \partial C$ . Take  $M$  as in Example 5.3, and let  $\alpha$  be a homotopically nontrivial embedded circle in  $M$  with a trivial normal bundle; the latter can be always arranged by replacing  $\alpha$  with its “square” in  $\pi_1(M)$ . Attach  $M \times [0, 1)$  to  $S^1 \times C$  by identifying the tubular neighborhood of  $\alpha$  in  $M \times \{0\}$  with  $S^1 \times \Delta$ . The result is an aspherical manifold  $N$  with compact boundary  $B := \partial N$ ; moreover, arguments of [Bel, Theorem 6.1] based on a strong version of Cantrell-Stallings hyperplane linearization theorem proved in [CKS] imply that the universal cover of  $\text{Int}(N)$  is diffeomorphic to  $\mathbb{R}^n$ . Since  $\pi_1(B)$  is an amalgamated product of  $\pi_1(M)$  and  $\mathbb{Z} \times \pi_1(\partial C)$  along  $\mathbb{Z}$ , its only nontrivial quotient in  $\mathcal{NP}_n$  is  $\pi_1(M)$ , so  $B$  cannot bound a manifold whose interior admits a complete metric of  $K \leq 0$  by Corollary 5.2.

## 6. PROOF OF THEOREM 4.1

*Proof.* Let  $\tilde{E}$  denote a connected  $H$ -invariant lift of  $E$  to  $\tilde{V}$ .

(1) The “only if” direction is trivial:  $B$  is incompressibility and compactness of  $B$  implies  $\text{cd}(H) = \dim(B)$ , so we focus on the “if” direction.

Compactness of  $B_1$  and the fact that  $H$  preserves a horoball implies that  $\tilde{B}_1$  is in the  $r$ -neighborhood of a horosphere  $H_1$  for some  $r$ . Let  $N_{r+1}(H_1)$  be the closed  $r+1$ -neighborhood of  $H_1$ .

We next show that  $\tilde{E}$  contains a component of  $\tilde{V} - N_{r+1}(H_1)$ . Otherwise, since  $\tilde{B}_1$  separates  $\tilde{V}$ , both components lie in  $\tilde{V} - \tilde{E}$ , so that  $\tilde{E} \subset N_{r+1}(H_1)$ . Since  $\text{cd}(H) = \dim(H_1)$  and  $H$  stabilizes  $H_1$ , the  $H$ -action on  $H_1$  is cocompact, and hence so is the  $H$ -action on  $N_{r+1}(H_1)$ . So  $E$  lies in a compact subset of  $V$ , which is a contradiction.

Denote the component of  $\tilde{V} - N_{r+1}(H_1)$  that lies in  $\tilde{E}$  by  $U_2$ ; then  $H_2 := \partial U_2$  is a horosphere concentric to  $H_1$ .

Let  $B_t$  be the fiber of the projection  $E \cong \partial E \times [1, \infty) \rightarrow [1, \infty)$  over  $\{t\}$ , and let  $\tilde{B}_t \subset \tilde{E}$  be the lift of  $B_t$ . Since  $B_t$  separates  $E$ , the distance from  $B_1$  and  $B_t$  is an increasing continuous function of  $t$ , and completeness of  $E$  implies that the function is unbounded. Hence the same is true for the distance from  $\tilde{B}_1$  and  $\tilde{B}_t$ , and therefore there is  $s$  such that the distance between  $\tilde{B}_1$  and  $\tilde{B}_s$  equals  $2r + 3$ . Note that  $U_2$  contains  $\tilde{B}_s$ .

Since  $H$  acts cocompactly on  $[1, s] \times \partial \tilde{E}$ , there is  $R$  such that  $\tilde{B}_s$  lies in the  $R$ -neighborhood of  $\tilde{B}_1$ , and hence in the  $R + r$ -neighborhood of  $H_1$ . Let  $H_3$  be a horosphere in  $U_2$  that bounds the  $R + r + 1$ -neighborhood of  $H_1$ , so that  $H_3$  is disjoint from  $[1, s] \times \partial \tilde{E}$ .

By construction the inclusion of  $\tilde{B}_s$  into  $\tilde{E}$ , which is a homotopy equivalence, factors through the contractible region between  $H_2$  and  $H_3$ , so  $\tilde{B}_s$  is contractible. It follows that  $B_s$  is aspherical and  $\pi_1$ -injectively embedded into  $V$ , and hence  $\partial N$  is incompressible in  $N$ .

(2) Let  $E_{<\varepsilon}$  denote the set of points of  $E$  with injectivity radius  $< \varepsilon$ . As  $\partial E$  is compact,  $\partial E \subset E_{<\varepsilon}$  if  $\varepsilon$  is small enough, which we assume henceforth. Let  $\tilde{E}_{<\varepsilon}$  be the preimage of  $E_{<\varepsilon}$  under the covering  $\tilde{E} \rightarrow E$ . The set  $\tilde{E}_{<\varepsilon}$  is open and locally convex for if  $d_\gamma(x) < \varepsilon$ , then  $d_\gamma(y) < \varepsilon$  for all  $y$  close to  $x$ , and convexity of  $d_\gamma$  implies that  $d_\gamma < \varepsilon$  on the segment  $[x, y]$ . Since  $\text{InjRad} \rightarrow 0$  on  $E$ , the  $H$ -action on  $\tilde{E} - \tilde{E}_{<\varepsilon}$  is cocompact, so since  $H$  stabilizes a horoball, there is  $r$  such that  $\tilde{E} - \tilde{E}_{<\varepsilon}$  lies in an  $r$ -neighborhood of a  $H$ -invariant horosphere, which we denote  $H_1$ . Let  $H_2$  be the horosphere that bounds the  $r + 2$ -neighborhood of  $H_1$  and lies in the horoball bounded by  $H_1$ . Let  $U_2$  be the horoball bounded by  $H_2$ . The distance between  $U_2$  and  $\tilde{E} - \tilde{E}_{<\varepsilon}$  is  $\geq 2$ .

Next show that  $\tilde{E}_{<\varepsilon}$  is convex and  $U_2 \subset \tilde{E}_{<\varepsilon}$ . Let  $Q$  be an arbitrary component of  $\tilde{E}_{<\varepsilon}$ . Thus  $Q$  is convex (as a connected, locally convex subset of a Hadamard manifold) and hence its boundary  $\partial Q \subset \tilde{E} - \tilde{E}_{<\varepsilon}$  is a topological, properly embedded submanifold that therefore separates  $\tilde{V}$ . Consider a ray  $\sigma$  that starts at a point  $\sigma(0) \in Q$  in the 1-neighborhood of  $\partial Q$ , and that ends in the center  $\sigma(\infty)$  of  $U_2$  at infinity. As  $\sigma(0)$  lies in the  $r + 1$ -neighborhood of  $H_1$ , the ray  $\sigma$  intersects  $H_2$ . If  $\gamma \in H$  whose displacement  $d_\gamma$  at  $\sigma(0)$  is  $< \varepsilon$ , then since  $H$  fixes  $\sigma(\infty)$ , nonpositivity of the curvature implies  $d_\gamma(\sigma(t)) < \varepsilon$  for all  $t$ , so  $\sigma \subset Q$ . As  $\partial Q$  separates  $\tilde{V}$  and is disjoint from  $U_2$ , we get  $U_2 \subset Q$ . Since  $Q$  is arbitrary and  $U_2$  is connected, we conclude that  $Q = \tilde{E}_{<\varepsilon}$ .

The rest of the proof works verbatim as in (1), which is a purely topological argument and in particular, the fact that  $U_2$  need not be a horoball in (1), while it is a horoball in (2) does not matter.

(3) If  $B$  is incompressible, then  $B$  is homotopy equivalent to a horosphere quotient, which is a closed manifold whose universal cover is diffeomorphic to  $\mathbb{R}^{n-1}$ , where  $n - 1 = \dim(B)$ . Thus the universal cover of  $B$  is properly homotopy equivalent to  $\mathbb{R}^{n-1}$ , and hence is simply-connected at infinity, and so homeomorphic to  $\mathbb{R}^{n-1}$  by [Sta62, Edw63, Wal65, Fre82].  $\square$

*Proof of Addendum 5.1.* The proof of (1) actually works when  $H_1$  is any codimension one properly embedded submanifold whose normal exponential map is a diffeomorphism. Then  $H_2, H_3$  are hypersurfaces equidistant to  $H_1$  that bound  $r + 1, r + R + 1$  neighborhoods of  $H_1$ , respectively.  $\square$

**Remark 6.1.** In fact, if  $B$  is incompressible in  $N$ , then  $B$  is h-cobordant to a horosphere quotient: the region between  $\tilde{B}_1$  and  $H_2$  projects to an h-cobordism embedded in  $E$ .

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